TOPOLOGICAL GRAVITY IN MINKOWSKI SPACE

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ABSTRACT. The two-category with three-manifolds as objects, h-cobordisms as morphisms, and diffeomorphisms of these as two-morphisms, is extremely rich; from the point of view of classical physics it defines a nontrivial topological model for general relativity.

A striking amount of work on pseudoisotopy theory [Hatcher, Waldhausen, Cohen-Carlsson-Goodwillie-Hsiang-Madsen . . .] can be formulated as a TQFT in this framework. The resulting theory is far from trivial even in the case of Minkowski space, when the relevant three-manifold is the standard sphere.

Topological gravity [18] extends Graeme Segal's ideas about conformal field theory to higher dimensions. It seems to be very interesting, even in **extremely** restricted geometric contexts:

§1 basic definitions

1.1 A **cobordism** $W : V_0 \to V_1$ between d-manifolds is a $D = d+1$-dimensional manifold W together with a distinguished diffeomorphism

$$\partial W \cong V_0^{op} \coprod V_1 \; ;$$

a diffeomorphism $\Phi : W \to W'$ of cobordisms will be assumed consistent with this boundary data.

$\mathbf{Cob}(V_0, V_1)$ is the category whose objects are such cobordisms, and whose morphisms are such diffeomorphisms. Gluing along the boundary defines a composition **functor**

$$\# : \mathbf{Cob}(V', V) \times \mathbf{Cob}(V, V'') \to \mathbf{Cob}(V, V'') \; .$$

The two-category with manifolds as objects and the categories **Cob** as morphisms is symmetric **monoidal** under disjoint union.

The categories **Cob** are topological **groupoids** (all morphisms are invertible), with classifying spaces

$$|\mathbf{Cob}(V_0, V_1)| = \coprod_{[W : V_0 \to V_1]} B\mathrm{Diff}(W \text{ rel } \partial) \; .$$

The **topological gravity** category has these objects as hom-spaces: it is a (symmetric monoidal) topological category.

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1.2 A **theory of topological gravity** is a representation of such a category in some simpler monoidal category, e.g. Hilbert spaces, or spectra.

The homotopy-to-geometric quotient map

$$B\mathrm{Diff} = \mathrm{Met} \times_{\mathrm{Diff}} E\mathrm{Diff} \to \mathrm{Met} \times_{\mathrm{Diff}} \mathrm{pt} = \mathrm{Met}/\mathrm{Diff}$$

defines a functor from the topological gravity category to a category with the spaces Met/Diff as morphism objects; these are the spaces of states in general relativity (and

$$g \mapsto \int R(g)\, d\mathrm{vol}_g$$

is a kind of Morse function upon them).

In Segal's conformal field theory, the corresponding objects are moduli spaces of (complex structures on) Riemann surfaces. Indeed if $W = \Sigma$ is a Riemann surface of genus > 1, its group of diffeomorphisms is homotopically discrete: the map

$$\mathrm{Diff}(\Sigma) \to \pi_0\, \mathrm{Diff}(\Sigma)$$

is a homotopy equivalence. The mapping class group acts with finite isotropy on Teichmüller space, so when $d = 1$ the homotopy-to-geometric quotient is close to a rational homology equivalence.

§2 examples

2.1 In recent work Galatius, Madsen, Tillmann and Weiss have identified the classifying space of the cobordism category of oriented d-manifolds in terms of a twisted desuspension $MT\mathrm{SO}(D)$ of the classifying space of the special orthogonal group. Their techniques extend more generally, to cobordism categories of manifolds with extra structure on their tangent bundle.

Three-manifolds under Spin cobordism have very interesting connections with the theory of even unimodular lattices [8,16], and the methods of [6] identify the classifying spectrum of this category with the desuspension of $B\mathrm{Spin}(4)$ by the vector bundle associated to the standard four-dimensional representation of the spin group. Because of well-known coincidences in low-dimensional geometry, $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, so we can identify its classifying space with the product of two copies of infinite-dimensional quaternionic projective space, and the vector bundle defined by the standard representation with the tensor product (over $\mathbb{H}$) of the resulting two canonical quaternionic line bundles $L_\pm$; thus

$$MT\mathrm{Spin}(4) \sim (\mathbb{H}P_\infty \times \mathbb{H}P_\infty)_+^{-L_+^{op} \otimes_{\mathbb{H}} L_-} .$$

The generators of $\pi_0 \Omega^\infty MT\mathrm{Spin}(4) \cong \mathbb{Z}^2$ can be identified with the signature and Euler characteristic, or alternately with the number of hyperbolic and E_8 factors in the middle-dimensional intersection form [2] of a spin cobordism.

2.2 There are other extremely interesting variant constructions in dimension four: contact three-manifolds under Spin^c cobordism define a natural context for Seiberg-Witten theory, while Lorentz cobordism [20] incorporates an arrow of time; but this note is concerned with **3-manifolds up to h-cobordism:**

Recall that $W : V_0 \to V_1$ is an h-cobordism if the two inclusions

$$V_0 \subset W,\ V_1 \subset W$$

are homotopy equivalences [17].

The **trivial** h-cobordism $W = V \times I$, where I is an interval, is an interesting example. In dimensions ≥ 5, the s-cobordism theorem classifies h-cobordisms by elements of the Whitehead group

$$\mathrm{Im}\,[\pm\pi_(V) \to K_1(\mathbb{Z}[\pi_1(V)])] := \mathrm{Wh}(\pi_1(V))\ ,$$

and there are invariants for **parametrized** h-cobordisms taking values in higher homotopy groups of certain pseudoisotopy spaces, which have been studied by Hatcher, Waldhausen, Igusa, ...

This category has a monoidal structure, but it is relatively trivial, so that it is natural to assume that the manifolds V are **connected**.

2.3 Here I will be concerned mostly with the case $V = S^3$: by Minkowski space I really mean the universal cover $S^3 \times \mathbb{R}$ of Penrose's (and others') conformal compactification $S^3 \times_{\pm 1} S^1$ of Minkowski space; this contains, in particular, a copy of Einstein's static universe [11]. Its time-like intervals define trivial h-cobordisms of S^3.

Note that there are lots of wild $S^3 \times \mathbb{R}$'s: remove a point from a fake $\mathbb{R}^4$. It would be very interesting to construct a semigroup of such things, under some kind of boundary gluing, as Segal did with topological annuli; current work of Gompf [10 §7, cf. also [3]] seems close to this. It is not clear at the moment if nontrivial smooth h-cobordisms of the three-sphere exist; the question is closely connected to the smooth four-dimensional Poincaré conjecture.

§3 double categories

3.1 Boundary value problems involve the interplay between diffeomorphisms of a manifold and diffeomorphisms of its boundary. Tillmann [21] suggests that **double** categories provide a natural framework for such questions. In this context, the primary objects are certain rectangular diagrams

$$\begin{tikzcd} W : \arrow[d, Rightarrow, "\Phi"'] & V_0 \arrow[r] \arrow[d, "\phi_0"] & V_1 \arrow[d, "\phi_1"] \\ W' : & V_0' \arrow[r] & V_1' \ . \end{tikzcd}$$

with cobordisms displayed horizontally, and diffeomorphisms (which preserve some boundary framing) presented vertically; these can be patched together in either direction. More recently, Getzler [7] has used manifolds, together with suitable (eg separating) codimension one submanifolds, to define morphisms in such contexts; this seems particularly suited to the *millefeuille* examples of Gompf, which (if I understand correctly) can be regarded as smooth h-cobordisms between topological, but not necessarily smooth, three-spheres.

3.2 In any case, the double category $\mathcal{D}$ of **trivial** h-cobordisms between **ordinary** three-spheres is already extremely interesting. I don't know how to associate a

topological category to a double category in general, but in this case pseudoisotopy theory defines an equivalence with the two-category

$$\coprod [\{V\}/\mathcal{C}(V)]$$

having manifolds V as its objects, and Cerf's group $\mathcal{C}(V)$ [13 §6.2] of pseudoisotopies (regarded as a category with one object) as its category of automorphisms:

These pseudoisotopies are diffeomorphisms of the cylinders $V \times I$, equal to the identity map on $V \times 0$. There is a fibration

$$\mathrm{Diff}(V \times I \text{ rel } \partial) \to \mathcal{C}(V) \to \mathrm{Diff}(V)$$

of groups, and **concordance**

$$\Phi, \Psi \mapsto \Phi \# (\phi_1 \times 1_I) \circ \Psi$$

of pseudoisotopies defines a homomorphism

$$\mathcal{C}(V) \times \mathcal{C}(V) \to \mathcal{C}(V) \; .$$

The classifying space $B\mathcal{C}(V)$ is thus a monoid, and the topological category associated to this rectification of $\mathcal{D}$ defines an *ad hoc* topologification (with one object for each V, and the topological monoid $B\mathcal{C}(V)$ for its space of endomorphisms). The classifying $B^2\mathcal{C}(V)$ space of **that** topological category is the totalization of the bisimplicial space defined by the category of trivial h-cobordisms of V.

3.3 There is a natural stabilization map from $B^2\mathcal{C}(V)$ to Waldhausen's ring spectrum $A(V)$. In the language of TQFT's, this defines a functor from the gravity category of trivial h-cobordisms of V to the category with $\{V\}$ as its object, and the group ring $S^0[\Omega \mathrm{Wh}^d(V)]$ as its endomorphism object. [The map from $\Omega B^2\mathcal{C}(V)$ to $\Omega A(V)$ factors through the space $\mathrm{HCobord}^d(V) \sim \Omega \mathrm{Wh}^d(V)$ of stabilized h-cobordisms of V [22].] This reveals Whitehead torsion (regarded as an element of $\mathbb{Z}[\mathrm{Wh}]$) as perhaps the primordial example of a TQFT!

Note that Cerf's maps define a fibration

$$B\mathrm{Diff}(V) \to B^2\mathrm{Diff}(V \times I \text{ rel } \partial) \to B^2\mathcal{C}(V)$$

which looks like a presentation of this *ad hoc* classifying space for a double category as a fibration

$$|\mathrm{Vertical}| \to |\mathrm{Horizontal}| \to |\mathrm{Double}|$$

built from classifying spaces for its component (vertical and horizontal) morphisms; but I don't know enough about double categories to guess if this might be an instance of something more general.

§4 about $A(S^n)$

4.1 Through the efforts of many researchers, a great deal is known about the algebraic K-theory of spaces; in particular, if X is simply connected (and of finite type) its A-theory can be calculated (at least p-locally [4 §1.3]) from the topological cyclic homology [14 §7.3.14] of $S^0[\Omega X]$.

Since this pretends to be a paper about physics, however, I will be content with some remarks about $A_*(X) \otimes \mathbb{Q}$, which is accessible in more elementary terms. [I want to record here my thanks to Bruce Williams and Bjorn Dundas for walking

me through a great deal of literature in this field, without suggesting that they bear any responsiblity for the excesses of this paper.]

4.2 In particular, old results [12] of Hsiang and Staffeldt imply that (when $n > 1$) the rationalization of $A(S^n)$ splits as a copy of $A(\mathrm{pt})_{\otimes}\mathbb{Q} \cong K^{\mathrm{alg}}(\mathbb{Z}) \otimes \mathbb{Q}$ and the suspension of what is essentially the (reduced) topological cyclic homology of S^n, which can be computed effectively as the abelianization of $\tilde{H}_*(\Omega S^n, \mathbb{Q})$ regarded as a graded Lie algebra; hence

$$\pi_*(\Omega \mathrm{Wh}^d(S^n)) \otimes \mathbb{Q} \cong K^{\mathrm{alg}}_{*+1}(\mathbb{Z}) \otimes \mathbb{Q} \oplus \tilde{H}_*(\Omega S^n, \mathbb{Q})_{\mathrm{ab}} \; .$$

The Whitehead product structure on a wedge of spheres is rationally free, so the graded Lie algebra structure has nontrivial commutators only when n is even. When $n = 2m + 1$ is odd, the rational homology is polynomial on a single generator x_{2m}; it follows that

$$A_{*+1}(S^3) \otimes \mathbb{Q} = \mathbb{Q}\langle \zeta_k, x_2^l \rangle$$

is spanned as a rational vector space by elements x_2^l of degree $2l$ and elements ζ_k of degree $4k$ corresponding to the odd zeta-values $\zeta(2k+1)$ which appear as regulators in Borel's calculations of $K^{\mathrm{alg}}_{4k+1}(\mathbb{Z}) \otimes \mathbb{Q}$.

This can be made more precise; when X is simply-connected then a reduced version $\Omega\widetilde{\mathrm{Wh}}(X)$ of loops on the Whitehead space is closely connected to a similarly reduced version $Q(\widetilde{LX}_{h\mathbb{T}})$ of (the infinite loopspace defined by) the suspension spectrum of the homotopy quotient (by its natural circle action) of the free loopspace of X.

4.3 The construction $Q = \Omega^\infty \Sigma^\infty$ sends a space to the infinite loopspace representing its suspension spectrum: this sends the rational homology of a space to its symmetric algebra. The cohomological invariants defined by the space of trivial h-cobordisms of the three-sphere thus resemble the 'big' phase spaces [9] studied in quantum cohomology: for example, the stable rational homology of the Riemann moduli space is essentially with the symmetric algebra on the homology of $\mathbb{C}P^\infty$, and is thus a polynomial algebra with one generator of each even degree.

The rational cohomology of the infinite loopspace $\Omega^{\infty+1}A(S^3)$ seems similar in many ways: it is again a polynomial algebra, now with one set of generators indexed by even integers, the other by integers $\equiv 0$ modulo four. Physicists see these symmetric algebras as Fock representations associated to certain polarized symplectic vector spaces. In our context this seems to be related to an 'almost' splitting

$$HC_{\mathrm{per}} \sim HC \oplus \mathrm{Hom}_{\mathbb{Q}[u]}(HC, \mathbb{Q}[u]) \; ,$$

of periodic cyclic homology [5] These representations have symmetries closely related to the Virasoro algebra, which lead [19] to interesting integrable systems.

This connection between 4D topological gravity and the equivariant free loopspace of the three-sphere resembles in many ways a purely mathematical instance of a phenomenon physicists [1] call 'holography', in which one physical model on the interior of a manifold is described by some other model on its boundary. Rather than proceed any further with speculations along these lines, I'd like to close by raising a mathematical question:

A trivial h-cobordism between three-spheres is an example of a four-dimensional spin cobordism; this defines a monoidal functor, and hence a morphism

$$\Sigma^{-1}A(S^3) \to MT\mathrm{Spin}(4) \sim (\mathbb{H}P_\infty \times \mathbb{H}P_\infty)_+^{-L_+^{op} \otimes_{\mathbb{H}} L_-}$$

of spectra. Could it possibly be nontrivial?

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